AN ANALYTICAL SOLUTION OF BIOT'S PROBLEM.

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In H. Laurent's Traité d'Analyse, tome 5, 1890, p. 110, we find the following problem due to Biot: Find a plane curve, such that all the luminous rays emanating from a fixed point, after two reflections on the curve, return to the fixed point. Laurent's solution is very simple, applying the common law of reflection, but it is wrong. Prof. M. Fijiwara has given a true solution to the problem in the Tohoku Mathematical Journal, volume 2, 1912, p. 149. I shall here give an analytical solution.

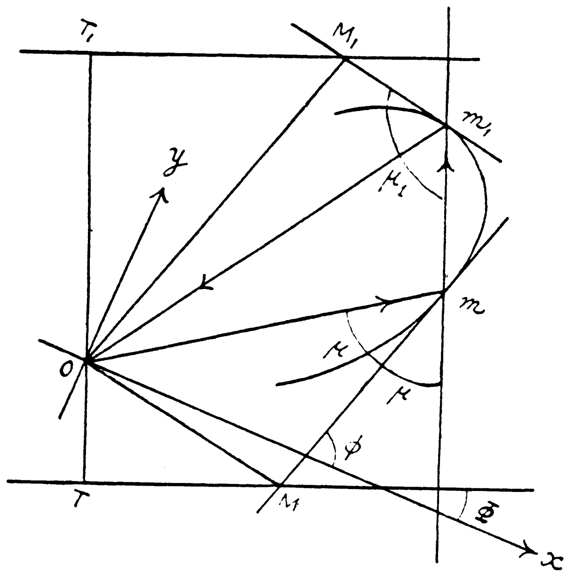

Fig. 1.

Let O be the source of light, and let Omm_1O be the path of a ray. Let μ be the angle which Om or mm_1 makes with the tangent mM at m, and let μ_1 be the angle which mm_1 or m_1O makes with the tangent m_1M_1 at m_1. Drop from O perpendiculars OM, OM_1 on mM, m_1M_1 respectively. Then M, M_1 lie on the pedal of the required curve.

Take O as origin and Ox, Oy as rectangular coördinate axes. Then the rectangular coördinates (x, y) of the point m are connected with the polar tangential coördinates (p, φ) of the same point by the relation

$$x \sin \varphi - y \cos \varphi = p,$$

μ being the perpendicular OM, and φ being the angle between Mm and Ox. This relation is the equation to the tangent Mm. Hence this equation and that got by differentiating it with respect to φ, i.e.,

$$x \cos \varphi + y \sin \varphi = dp/d\varphi, = p' \text{ say},$$

give the rectangular coördinates (x, y) of the point m in terms of the polar tangential coördinates (p, φ). Thus

$$x = p \sin \varphi + p' \cos \varphi,$$

$$y = - p \cos \varphi + p' \sin \varphi.$$

The equation of OM is

$$x \cos \varphi + y \sin \varphi = 0.$$

Therefore the rectangular coördinates (X, Y) of the point M are

$$X = p \sin \varphi, \qquad Y = - p \cos \varphi.$$

Hence the tangent to the pedal of the required curve, i.e., the locus of M, at M has the direction given by

$$\frac{dY}{dX} = - \frac{d(p \cos \varphi)}{d(p \sin \varphi)} = - \frac{p' \cos \varphi - p \sin \varphi}{p' \sin \varphi + p \cos \varphi}, = \tan \Phi \text{ say}.$$

Now by the relation

$$x \cos \varphi + y \sin \varphi = p',$$

mM is equal to p', so that

$$\tan \mu = p/p'.$$

Hence the angular coefficient of mm_1 is given by

$$\tan (\varphi + \mu) = \frac{p' \sin \varphi + p \cos \varphi}{p' \cos \varphi - p \sin \varphi}.$$

Therefore

$$\frac{\pi}{2} + \Phi = \varphi + \mu,$$

i.e., mm_1 and TM make right angles.

Similarly m_1m and T_1M_1, tangent to the pedal at M_1, make right angles. Therefore TM and T_1M_1 are parallel, and the perpendiculars OT and OT_1 dropped from O on TM and T_1M_1 respectively lie on one and the same straight line. Denote the length of OT by P, so that the polar tangential coördinates of the point M are (P, Φ), while its rectangular coördinates are (X, Y).

The equation to mm_1, regarded as passing through m, is

$$\eta + p \cos \varphi - p' \sin \varphi = \tan (\varphi + \mu) \cdot (\xi - p \sin \varphi - p' \cos \varphi),$$

ξ, η being current coördinates. But

$$
\begin{aligned}
-\tan(\varphi+\mu)\cdot(p\sin\varphi+p'\cos\varphi)-p\cos\varphi+p'\sin\varphi \\
&= -\frac{2pp'}{p'\cos\varphi-p\sin\varphi} \\
&= -2pp'(p^2+p'^2)^{-\frac{1}{2}}(\cos\mu\cos\varphi-\sin\mu\sin\varphi)^{-1} \\
&= -2pp'(p^2+p'^2)^{-\frac{1}{2}}\{\cos(\varphi+\mu)\}^{-1} \\
&= 2pp'(p^2+p'^2)^{-\frac{1}{2}}(\sin\Phi)^{-1}.
\end{aligned}
$$

Similarly, from the equation to mm_1, regarded as passing through m_1, we have

$$
-\tan(\varphi_1+\mu_1)\cdot(p_1\sin\varphi_1+p_1'\cos\varphi_1)-p_1\cos\varphi_1+p_1'\sin\varphi_1 = 2p_1p_1'(p_1^2+p_1'^2)^{-\frac{1}{2}}(\sin\Phi_1)^{-1}.
$$

These two expressions must be equal, since they come from the equations to the same straight line mm_1. Hence

$$pp'(p^2+p'^2)^{-\frac{1}{2}}(\sin\Phi)^{-1} = p_1p_1'(p_1^2+p_1'^2)^{-\frac{1}{2}}(\sin\Phi_1)^{-1}.$$

But

$$\Phi_1 = \pi+\Phi.$$

Therefore

$$pp'(p^2+p'^2)^{-\frac{1}{2}} + p_1p_1'(p_1^2+p_1'^2)^{-\frac{1}{2}} = 0.$$

Now by a well-known theorem, OM is the mean proportional between OT and OM.* Hence

$$P = p^2(p^2+p'^2)^{-\frac{1}{2}}.$$

Therefore

$$\frac{dP}{d\Phi} = \frac{p^3p'+2pp'^3-p^2p'p''}{(p^2+p'^2)^{3/2}}\cdot\frac{d\varphi}{d\Phi}.$$

But from the relation

$$\varphi+\mu = \varphi+\tan^{-1}\frac{p}{p'} = \frac{\pi}{2}+\Phi,$$

we have

$$\frac{d\varphi}{d\Phi} = \frac{p^2+p'^2}{p^2+2p'^2-pp''}.$$

Therefore

$$\frac{dP}{d\Phi} = \frac{pp'}{(p^2+p'^2)^{1/2}}.$$

Similarly

$$\frac{dP_1}{d\Phi_1} = \frac{p_1p_1'}{(p_1^2+p_1'^2)^{1/2}}.$$

* See, e.g., Williamson's Differential Calculus, 1892, p. 228.

But the sum of the right-hand members is equal to zero as has been shown above. Therefore

$$\frac{dP}{d\Phi} + \frac{dP_1}{d\Phi_1} = 0,$$

i.e.,

$$P'(\Phi) + P'(\Phi + \pi) = 0.$$

Integrating with respect to Φ,

$$P(\Phi) + P(\Phi + \pi) = \text{const.}$$

Hence *the pedal of the required curve is a curve of constant breadth.*

The converse can be similarly treated. *Singular solutions* are got by putting

$$\tan(\varphi + \mu) = 0 \qquad \text{or} \qquad = \infty,$$

i.e.,

$$p' \sin\varphi + p\cos\varphi = 0, \qquad \text{or} \qquad p'\cos\varphi - p\sin\varphi = 0,$$

i.e.,

$$p = \frac{\text{const.}}{\sin\varphi}, \qquad \text{or} \qquad p = \frac{\text{const.}}{\cos\varphi}.$$

Therefore the oval included by two confocal parabolas having the same axis, but in opposite senses, is the required curve.

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